General Topology and Applications

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Hemant Kumar Pathak

General Topology and Applications

A First Course

Sciengér Draft

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Graduate Texts in Mathematics 2
General Topology and Applications
First Edition 2022
ISBN 978-81-956379-8-0 ISBN 978-81-956379-2-8 (eBook)
Mathematics Subject Classification (2020): 54-XX,54Axx,54Bxx,54Cxx,54Dxx,54Exx,54Hxx.

Design works: G.K.Graphics, Bhilai
Laser Type Setting: In-house, Scienger Draft, Bhilai
Printed in India

Printed on acid-free paper

This Scienger imprint is published by the registered proprietary firm Scienger Draft, Bhilai, India

The registered firm address is: Block 47(A), Plot No. 1, M. Nehru Nagar (West), Bhilai, Chhattisgarh, India

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Dedicated to

COVID-19 Warriors
who lost their lives during Ist and 2nd wave of
pandemic in honor of their great service to mankind
and
lakhs of hard working migrant labours of India who faced a very difficult time in the first lock-down period during May-June, 2020; many of them lost their lives in a journey to return their native places, many more were waiting to start journey in the second wave with great anxiety, hungry & thirsty all around

Hemant Kumar Pathak

Preface

This book entitled with "General Topology and Applications" is designed as a text book for two semesters course usually called "Point Set Topology" or "Metric Topology". This text is designed for the first semester course at the lower undergraduate level with the title 'Introduction to General Topology' covers chaps. 2 to 6 and the second semester course with the title 'Advances in General Topology' covering chaps. 7 to 11. It is accessible to junior mathematics majors who have studied multivariable calculus.

Pre-Requisites

The essential prerequisites for reading this book are quite minimal: not much more than a stiff course in analysis, advanced calculus and abstract algebra. We have included every topic that we think essential for the training of topology, and to enhance the understanding of the subject we have also gone down a number of bypaths. We have given complete definitions well explained by suitable examples, explanations, and proofs throughout the text, so that the book should be usable for indivisible study as well as for a course work.

General Topology is an important branch of mathematics having immense interest in its own right, and it also serves to lay the foundations for future study in various branches of mathematics. It deals with points and sets in topological spaces, and continuous functions between topological spaces. It also deals with basic properties of topological spaces, such as compactness and connectedness, that depend only on the topology. It is the purest form of topology and provides a basic language for the rest of mathematics, relying on the basic definitions of open sets and topological spaces. The important feature of general topology is that the very wide use of its methods in many of the other branches of modern mathematics such as those dealing with functional analysis, partial differential equations, classical mechanics, theoretical physics, general theory of relativity, mathematical economics, biology, etc. Topology is, indeed, an abstract mathematical theory that connects different branches, carries the potential of new development of mathematics. Nowadays topology has become a powerful instrument of mathematical research, and its language acquired importance.

The contents of the book has been divided into thirteen chapters, is devoted to the subject commonly called *general topology* or *point set topology*. Chapters 2-11 deal with the body of material that is included in any introductory topology course worthy of the name. This may be considered the "irreducible core" of the subject, treating as it does set theory, topological spaces, connectedness, compactness (through compactness of finite products), and the countability and separation axioms. The remaining two chapters; that is, chaps. 10 and 11 explore additional topics; they are essentially independent of one another, depending on only the core material of Chapters 2-9. The instructor may take them up in any order he or she chooses.

Chapter 1 begins with an introduction to the basic concepts of topology, namely, set theory, logic, relations, basic results of analysis, fields of reals, complex numbers and quaternion $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ and their properties. Especially 'cardinality of sets', 'continuum hypothesis' and the 'Schauder-Berstein theorem' are discussed in details. Indeed, the first chapter is strictly preparatory and may be assigned as a self-study.

Chapter 2 provides a motivation for topology through geometry by introducing the notions of topology and its base and subbase. The key first step in the theory is to extend the concepts of open sets, closure, interior and boundary of a set, subspace from metric space to topological space. An account of continuous functions, the notions of product topology and quotient topology are also discussed.

Chapter 3 studies countability axioms and separability. Connectedness, components, local connectedness and path-connectedness are the main topics for our study and are discussed in great details.

Chapter 4 deals with compact spaces and the Tychonoff theorem. In this chapter, we focus our studies on compactness, countable compactness, sequential compactness, local connectedness and limit point compactness. An account of the Alexander subbase theorem is also given.

Chapter 5 deals with separation axioms. In fact, $T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$ and $T_5$are studies.

Chapter 6 deals with compactification, In this chapter we discuss Alexandroff's one point compactification and the Stone-Čech compactification theorem in great details.

Chapter 7 introduces the concepts of product topology, coproduct and Moore-Smith convergence. The object of the present chapter is to discuss in details the Tychonoff product topology, productive, countably productive properties, introduces the concepts of nets and filters. As an application of filters, we have discussed that compactness is a productive property.

Chapter 8 introduces the notions of embedding, metrizability, local compactness and paracompact spaces. Uryshon metrization theorem, Nagata-Smirnov metrization theorem and Smirnov metrization theorem are discussed in great details.

Chapter 9 introduces the concepts of nets and filters.

Chapter 10 deals with uniform spaces. In this chapter we discuss uniform continuity, product uniformity, metrization, uniformity via pseudo-metrices, uniform subspaces and proximity spaces. Completeness and compactness in uniform spaces are also discussed.

Chapter 11 studies complete metric spaces, gauge spaces, space filling curves and function spaces. In this chapter we define topologies and uniformities for the set of continuous functions and prove compactness, completeness and continuity properties for the resulting spaces. Finally, Ascoli's theorem is discussed.

Chapter 12 introduces the notions of manifold and its geometry. In particular, imbedding of manifolds and some application to geometry of the universe is also discussed.

Chapter 13 introduces the notion of Baire space and Baire's category theorem is discussed. In this chapter we also introduce dimension theory.

This book evolved from classes taught by the author at Kalyan PG College, Bhilai and Pt. Ravishankar Shukla University, Raipur, India. Furthermore, the book is self-contained and the presentation is detailed, to avoid irritating readers by frequent references to details in other books. The examples are simple, to make the book teachable. We hope that this book will be extremely useful to students having no background of 'topological properties'.

Suggested Course Outlines

Evidently, this book has been arranged according to the course content of several leading universities of the world. No doubt, several topics are independent of one another, so it is profitable to advise the reader what should be read before a particular chapter.

Beyond any doubt, the book contains too much material that can be covered in a one-year course, but there is considerable flexibility for individual course design. Chapters $2-11$ are suitable for a full-year course in general topology at the lower undergraduate level. For a one-year graduate course, we suggest Chaps. $2-6$ and $7-11$. The subject matter of Chaps. 12 and 13 can be studied just after finishing the core part with the adaptability of turning to materials of Chaps. 12 and 13 as and when needed. Keeping in view the study of the subject in the framework of two semesters course, the dependencies of chapters with only exception of chaps. 12 and 13 can roughly be divided as follows:

First Semester Course

↓ Chapter 1 → 2
↓ Chapter 3
↓ Chapter 4
↓ Chapter 5
↓ Chapter 6

Second Semester Course

↓ Chapter 7
↓ Chapter 8
↓ Chapter 9
↓ Chapter 10
↓ Chapter 11
↓ Chapters 12- 13

There are many people who deserve my gratitude in connection with the writing of this book. The author is highly indebted to many of his former teachers, colleagues, and students who directly or indirectly helped him in preparing this book. In particular, we extend very special cordial thanks to our colleagues and research collaborators Professors Satya Deo, V. Kannan, B. K. Sharma, M. Imdad, P. Veeramani, B. S. Thakur, D. R. Sahu, Hemant K. Nashine, P. P. Murthy all from India, Professors N. Shahzad, N. Hussain, Reny George, R. Rajagopalan and Fahad.S.Alshammari from Saudi Arabia, Professor S.S.Chang from China, Professors Shin Min Kang, J. S. Ume, B.S. Lee from S. Korea, Professor S. N. Mishra from S.

Africa, Professor Brian Fisher from England, Professors Billy E. Rhoades, Ravi P. Agarwal, and Gerald Jungck from the USA, Professors Yeol Je Cho, Shin Min Kang from S. Korea who gave much to the beauty and power of mathematics. It is a pleasure to acknowledge the great help and technical support given us by the publisher, in their rapid and meticulous publication of the work.

May 15, 2022 H. K. Pathak
(On the occasion of International Day of Families)

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Glossary of Symbols

$\mathbb{N}$ or $\mathbb{Z}_+$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{C}_\infty$	extended complex plane $\mathbb{C}\cup\{\infty\}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}^n$	n-dimensional unitary space
$\mathbb{H}^n$	The set of all n-tuples $(q_1, q_2, \cdots, q_n)$ of quaternions
$\in$	belongs to
$\notin$	does not belong to
$\|, :$	such that
$\subset$	subset
$\cup$	union
$\cap$	intersection
$-, \setminus$	difference
I, Λ	index set
$\emptyset$	empty set
$\prod_{i=1}^{n}$	product, i ranging from 1 to n
(a, b)	ordered pair
int A, A°	interior of A
∂A, A^b	boundary of A
$A \times B$	cartesian product of sets A and B
$\mathcal{D}(f)$	domain of f
$\mathcal{R}(f)$	range of f
aRb	a is R-related to b
$\forall$	for all
$\Rightarrow$	implies
$f : X \to Y$	f is a function from X to Y
$f(A)$	image of A under f
$f^{-1}(B)$	inverse image of B under f
i_X	identity map on X
$i : B \to A$	inclusion function
$f\|_A$	restriction f to A
$(x_n), \{x_n\}$	sequence whose n^{th} term is x_n

$\mathscr{P}(X)$	collection of all subsets of X
(a,b)	open interval $\{x : a < x < b\}$
$[a,b]$	closed interval $\{x : a \leq x \leq b\}$
f^{-1}	inverse mapping
$f \circ g$	composition of mappings f and g
d, ρ	metrics
$\bar{\rho}$	uniform metric
$B(x,r)$	open ball centered at x and of radius r
$\overline{B}(x,r)$	closed ball centered at x and of radius r
$\mathcal{T}$	topology
$\mathcal{T}_d$	topology induced by metric d
$\mathcal{T}_p$	topology of pointwise convergence
$\mathcal{T}_c$	topology of compact convergence (or compact-open topology)
$\mathcal{B}$	base for a topology
$\mathscr{U}, \mathscr{V}$	uniformities
$\mathscr{B}$	base for a uniformity
$\mathscr{T}_u$	topology induced by uniformity $\mathscr{U}$
$\pi_n(X, x_0)$	n-dimensional homotopy group
$\mathbb{R}P^n$	real n-dimensional projective space
$\sup S$	supremum of S
$\inf S$	infimum of S
$\lvert z \rvert$	modulus of z
$\bar{z}$	conjugate of z
$P_n(z)$	polynomial of degree n
$\exp(z)$	exponential function of z
$(x : P(x))$	set of all x such that $P(x)$
Ω	The first (or least) uncountable ordinal number
$\prod_{\alpha \in J} X_\alpha$	direct product of spaces X_α