Functional Analysis and Applications

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Functional Analysis and Applications

A First Course

Sciencer Draft

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The volumes in this series by Scienger Draft are mainly designed as graduate studies texts in Mathematics and its Applications including the research monographs in specific areas. However, they will also be suitable for supplementary course readings. These texts will make ideal independent study resources for students as well as the teaching faculty. We hope that the wide coverage of the material will make them an ideal acquisition for all academic institutions and individuals.

The Inaugural Volume on Functional Analysis and Applications by Hemant Kumar Pathak is dedicated to Late Professor Billy Eugene Rhoades, popularly known as Billy Rhoades in Mathematical circles from Indiana University, Bloomington who passed away on 7 May 2021. He made significant contribution in analysis and had impact on our own work(ref. Academic Journey of Billy E. Rhoades by S. N. Mishra, Indian J. Math. Volume 56(1), 014).

We look for your support for this venture.

Swami N. Mishra, Editor
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Preface

This book entitled with "Functional Analysis and Applications" is designed as a text book for two semester courses usually called "Introductory Functional Analysis" and "Advanced Functional Analysis". The first course is customarily offered at the senior undergraduate course while the second course is offered at the beginning of graduate course. It is presumed that students have some prior knowledge of analysis, linear algebra, point set topology and measure theory. In particular, basic knowledge of differential and integration theory, real and complex analysis, metric topology, the geometry of the linear space, the linear operators on the space, the existence of a linear space with a topology or two (or more), measurable functions and measure spaces.

The subject functional analysis is of immense interest in its own right, and it also serves to lay the foundations for future study in mathematical analysis, in harmonic analysis, in linear algebra, in geometry, physics. In this book we present fundamental material that will also be of interest to scientists, in particular, computer scientists, physicists, mechanical engineers, and electrical engineers.

The word Course in the title of this book has two meanings. The first is obvious. This book was meant as a text for the senior undergraduate course and the beginning of graduate course in functional analysis. The second meaning is that the book attempts to take an excursion through many of the territories that comprises functional analysis. This excursion will give an opportunity to the reader a choice of several tours in the territories of functional analysis. This will offer the reader an opportunity to get more involved in the subject to see some applications to other parts of mathematics.

Pre-requisites

A good grasp of linear algebra, real and complex analysis, and advanced calculus may be enough for the first course while introductory knowledge of measure theory may be necessary for the second course. Introductory knowledge of topology is already covered any course on real analysis under the topology of real line and may be enough to proceed with. However, for an easy reading and to avoid irritations of referring to other texts books over and again, all the fundamental concepts are quickly captured in Chapter 1.

The contents of the book have been divided into sixteen chapters, is devoted to the subject commonly called *functional analysis*. Chapters 2-13 deal with the body of material that is included in any introductory functional analysis course worthy of the name. This may be considered the "irreducible core" of the subject, treating as it does normed linear spaces, spaces of bounded linear functions, Hilbert spaces, linear operators, Banach algebra, differentiation and integration in Banach spaces, variational methods and optimization problems, operator theory, nonlinear operators, spectral theory, approximation theory and optimization and fixed point theory. The remaining three chapters; that is, chaps. 14-16 explore additional topics; they are essentially independent of one another, depending on only the core material of Chapters 2-13. Instructors may take them up in any order they prefer.

Chapter 1 begins with an introduction to the basic concepts of analysis and topology, in particular, the notions of a metric space and its variants, topological spaces, linear spaces and some basic tools of measure theory.

Chapter 2 is devoted to a generalization of n-dimensional Euclidean space, namely $\mathbb{R}^n$ as well as n-unitary space $\mathbb{C}^n$, known as Banach space. In this chapter we study spaces of sequences and spaces of different classes of functions such as spaces of continuous differential integrable functions. The properties of set of all linear/bounded operators or mappings have been studied. Geometrical and topological properties of Banach space and its general case normed linear space are presented.

Chapter 3 deals with five fundamental results of functional analysis, namely Hahn-Banach theorem, Banach-Alaoglu theorem, uniform boundedness principle, open mapping theorem, and closed graph theorem. Indeed, these theorems have played significant role in approximation and optimization theory, Fourier analysis, numerical analysis, control theory, mechanics, mathematical economics, and differential and partial differential equations.

Chapter 4 is devoted to the study of the dual space of some familiar Banach spaces of sequences and functions. The concepts of weak convergence of sequences of elements in a normed linear space and the weak* convergence of sequences of functionals are introduced. We also discuss the notions of strong and weak topologies and observe that the concept of weak topology in the setting of a normed linear space provides an interesting criteria for reflexivity.

Chapter 5 studies a special class of Banach space in which the underlying vector space is equipped with a structure, called an inner product providing the generalization of geometrical concepts like angle and orthogonality between two vectors. The inner product is, indeed, a generalization of the dot product of vector calculus. In the sequel, we study Hilbert space with its beautiful geometrical properties and Hilbert space method. It is, indeed, a powerful tool to tackle problems of diverse fields of classical mathematics like linear equations, approximation theory, variational methods, ordinary differential equations.

Chapter 6 introduces the notion of Banach algebra; its theory constitute a field of study in which a wide variety of mathematical ideas meet in significant contact. In the sequel, we discuss some useful applications of Banach algebra.

Chapter 7 is is devoted to the study of differentiation and integration of operators defined on a Banach space into another Banach space. In this chapter we give three crucial theorems, namely Taylor's theorem, inverse function, and implicit function theorems. These theorems have played significant role in approximation theory, Fourier analysis, numerical analysis, optimization, differential and partial differential equations. We discuss an important concept of nonlinear analysis, namely subdifferential of convex functionals. Finally, we discuss about integration in Banach spaces.

Chapter 8 introduces certain topics in the elementary theory of Hilbert spaces which lead directly to abstract variational or weak formulation of boundary value problems. We also discuss the notion of optimization of a functional defined on a normed space by Banach space and Hilbert space.

Chapter 9 is devoted to the study of approximation and optimization. We present a generalized form of Weierstrass theorem commonly known as Stone-Weierstrass theorem. This theorem relates to continuous functions defined on compact Hausdorff spaces. and has become an indispensable tool in topology and modern analysis.

Chapter 10 introduces the notions of closed operators, compact linear operators, finite-rank operators, positive semidefinite operators as well as strongly positive operators in Banach spaces. The important role of closed linear operators in the theory of unbounded operators is studied.

Chapter 11 deals with certain non-linear operators which arise naturally in differential geometry and mathematical physics.

Chapter 12 introduces the notions of the resolvent, the spectrum set of a bounded linear operator and projection operator. The notion of projection operator is used to reformulate another form of the spectral theorem which is of theoretical importance because it generalizes to non-compact operators more easily. Using spectral theorem to "Fredholm alternative", we obtain the existence and uniqueness of solutions of linear operator equations.

Chapter 13 discusses mainly Banach's fixed point theorem, Brouwer's fixed point theorem, Schauder fixed point theorem and applications. Fixed point results in ordered Banach spaces are also discussed.

Chapter 14 deals with mathematical models of real-world problems known as variational inequalities introduced systematically by J.L. Lions and S. Stampachia in early seventies. Modeling, discretization, algorithms, and visualization of solutions along with updated references are presented.

Chapter 15 discusses frame and basis theory in Hilbert spaces. The essence of this chapter is to show that every element of an inner product space can be expressed as a linear combination of elements in a given frame where linear independence is not required.

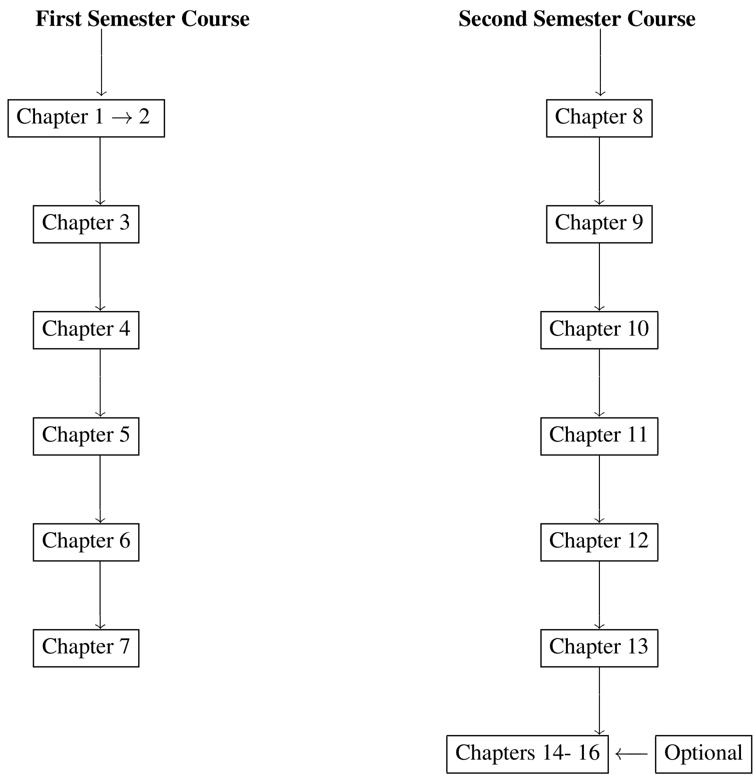
Chapter 16 discusses frames and bases in Banach spaces. Retro Banach frame, Λ-Banach frame, fusion Banach frame, near exact Banach frame, $\mathcal{F}$-Banach frame, and their applications are discussed.

This book evolved from classes taught by the author at Kalyan PG College, Bhilai and Pt. Ravishankar Shukla University, Raipur, India. Moreover, the book is self-contained and the presentation is detailed, to avoid irritating readers by frequent references to details in other books. The examples are simple, to make the book teachable. We hope that this book will be extremely useful to students having no background of 'topological properties'.

Suggested Course Outlines

Evidently, this book has been arranged according to the course content of several leading universities of the world. No doubt, several topics are independent of one another, so it is profitable to advise the reader what should be read before a particular chapter.

Beyond any doubt, the book contains too much material that can be covered in a one-year course, but there is considerable flexibility for individual course design. Chapters 2 – 13 are suitable for a full-year course in functional analysis at the lower undergraduate level. For a one-year graduate course, we suggest Chaps. 2 – 7 and 8 – 13. The subject matter of Chaps. 14-16 can be studied just after finishing the core part with the adaptability of turning to materials of Chaps. 14-16 as and when needed. Keeping in view the study of the subject in the framework of two semesters course, the dependencies of chapters with only exception of chaps. 13-16 can roughly be divided as follows:



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October 2, 2021 H. K. Pathak

(On the occasion of Mahatma Gandhi's 153rd Birth Anniversary)

Contents

About the Author

Hemant Kumar Pathak has been professor and Head of the School of Studies in Mathematics at the Pt. Ravishankar Shukla University, Raipur, India. He has been the dean of science, member of standing committee, chairman, UFM Committee, director of the Center for Basic Sciences, and director of the Human Resource Development Centre at the same university. He has previously worked at Kalyan P.G. College, Bhilai Nagar, and the Government Postgraduate College, Dhamtari, India. He has also been Vice Chancellor of Bharti Vishwavidyalaya, Durg. He earned his Ph.D. from Pt. Ravishankar Shukla University, Raipur in 1988.

Professor Pathak was awarded the "Distinguished Service Award 2011" by the Vijnana Parishad of India. With over 41 years of teaching and research experience, he has published three books, *Theory of Functions of Complex Variable* (Taylor & Fransis Group), *An Introduction to Nonlinear Analysis and Fixed Point Theory* (Springer Nature), *Complex Analysis and Applications* (Springer Nature), and more than 250 research papers in leading international journals of repute on approximation theory, operator theory, integration theory, fixed point theory, number theory, cryptography, summability theory, fuzzy set theory and Banach frames.

Professor Pathak currently serve on the editorial boards of American Journal of Computational and Applied Mathematics, Fixed Point Theory and Algorithms for Sciences and Engineering (Springer Nature), and the Journal of Modern Methods in Numerical Mathematics, and as a reviewer for the Mathematical Review of the American Mathematical Society. In addition, he is a life member of the Allahabad Mathematical Society, Bharata Ganita parishad, the Vijana Parishad of India, Calcutta Mathematical Society, and National Academy of Mathematics.

Glossary of Symbols

$\mathbb{N}$ or $\mathbb{Z}_+$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{C}_\infty$	extended complex plane $\mathbb{C} \cup \{\infty\}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}^n$	n-dimensional unitary space
$\mathbb{H}^n$	The set of all n-tuples $(q_1, q_2, \cdots, q_n)$ of quaternions
$\in$	belongs to
$\notin$	does not belong to
$\|, :$	such that
$\subset$	subset
$\cup$	union
$\cap$	intersection
$-, \setminus$	difference
I, Λ	index set
$\emptyset$	empty set
$\prod_{i=1}^{n}$	product, i ranging from 1 to n
(a,b)	ordered pair
int A, A°	interior of A
∂A, A^b	boundary of A
$A \times B$	cartesian product of sets A and B
$\mathcal{D}(f)$	domain of f
$\mathcal{R}(f)$	range of f
aRb	a is R-related to b
$\forall$	for all
$\Rightarrow$	implies
$f : X \to Y$	f is a function from X to Y
$f(A)$	image of A under f
$f^{-1}(B)$	inverse image of B under f
i_X	identity map on X
$i : B \to A$	inclusion function
$f\|_A$	restriction f to A
$(x_n), \{x_n\}$	sequence whose n^{th} term is x_n

$\mathscr{P}(X)$	collection of all subsets of X
(a,b)	open interval $\{x : a < x < b\}$
$[a,b]$	closed interval $\{x : a \leq x \leq b\}$
f^{-1}	inverse mapping
$f \circ g$	composition of mappings f and g
d, ρ	metrics
$\bar{\rho}$	uniform metric
$B(x,r)$	open ball centered at x and of radius r
$\overline{B}(x,r)$	closed ball centered at x and of radius r
$\mathcal{T}$	topology
$\mathcal{T}_d$	topology induced by metric d
$\mathcal{T}_p$	topology of pointwise convergence
$\mathcal{T}_c$	topology of compact convergence (or compact-open topology)
$\mathcal{B}$	base for a topology
$\pi_n(X, x_0)$	n-dimensional homotopy group
$\mathbb{R}P^n$	real n-dimensional projective space
$\sup S$	supremum of S
$\inf S$	infimum of S
$\|z\|$	modulus of z
$\bar{z}$	conjugate of z
$P_n(z)$	polynomial of degree n
$\exp(z)$	exponential function of z
$(x : P(x))$	set of all x such that $P(x)$
$\bigoplus_{\alpha \in J}$	direct sum of groups $G_\alpha,\ \alpha \in J$
$\prod_{\alpha \in J} X_\alpha$	direct product of spaces X_α
$\prod^*_{\alpha \in J} G_\alpha$	direct product of free groups G_α