Algebraic Topology and Applications

Graduate Texts in Mathematics Series

Graduate Texts in Mathematics series publishes high-quality textbooks, reference books, research based monographs, lecture notes and contributed volumes. Undergraduate and graduate students of mathematics, research scholars, and teachers would find this book series useful. Keeping in view the needs of the students a large number of examples have been given with enough details, followed by well-graded set of unsolved problems in exercises. The volumes are carefully written as teaching aids, highlight characteristic features of the theory and learning based observation at the end of each component of the theory. The books in this series are published by Scienger Draft, Bhilai, India.

More information about this series at http://www.scienger.com (or https://scienger.com)

Hemant Kumar Pathak

Algebraic Topology and Applications

A First Course

Sciencer Draft

Hemant Kumar Pathak
Formerly Professor of Mathematics
School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur, Chhattisgarh
India

Graduate Texts in Mathematics 3
Algebraic Topology and Applications
First Edition 2022
ISBN 978-81-956379-9-7 ISBN 978-81-956379-3-5 (eBook)
Mathematics Subject Classification (2020): 55-XX,55Mxx,55Pxx,55Qxx,55Rxx,55Uxx,57Kxx,57Sxx.

Design works: G.K.Graphics, Bhilai
Laser Type Setting: In-house, Scienger Draft, Bhilai
Printed and bound in India

Printed on acid-free paper

This Scienger imprint is published by the registered proprietary firm Scienger Draft, Bhilai, India

The registered firm address is: Block 47(A), Plot No. 1, M. Nehru Nagar (West), Bhilai, Chhattisgarh, India

Visit us at: https://scienger.com

Dedicated to

visionary leadership, scientists, and COVID-19 worriers
in honor of their great services to mankind, who are behind the vaccination
drive all over the world saving lives of billions of people from corona virus
and
my better half
Smt. Annapurna Pathak, who kept me safe in this pandemic

Hemant Kumar Pathak

Preface

Algebraic topology is a most flourishing and productive branch of mathematics that uses tools from abstract algebra to study topological spaces and one creates topological invariants that are algebraic in nature. The basic goal of studding algebraic topology is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence. Looking over different approaches to study topology, one can see that there are mainly five core branches of topology , viz., point set topology, geometric topology, algebraic topology, combinatorial topology, and differential topology, Algebraic topology is, in fact, an important branch of topology having several connections with many areas of modern mathematics. The essence of algebraic topology lies on the fact that it encompasses algebra and topology in a single thread resulting enormous applications in various domains of applied sciences. Its growth and influence, particularly since the early forties of the twentieth century, has been remarkably high. Presently, it is being taught in many universities at the upper undergraduate level in BS/BSc Hons/ MS/MSc or beginning PhD level as a compulsory or as an elective course. It is best suited for those who have already had an introductory course in topology as well as in algebra. There are several excellent books, starting with the first monograph 'Foundations of Algebraic Topology' by S. Eilenberg and N.E. Steenrod, 'A Basic Course in Algebraic Topology' by W.S. Massey, 'Algebraic Topology by W. Fulton and many more. Any one of these can be prescribed as a textbook for a first course on algebraic topology by making proper selections. However, there is no general agreement on what should be the 'first course' in this subject. Experience suggests that a comprehensive coverage of the topology of simplicial complexes, simplicial homology of polyhedra, fundamental groups, covering spaces and some of their classical applications like invariance of dimension of Euclidean spaces, Brouwer's Fixed Point Theorem, etc. are the essential minimum which must find a place in a beginning course on algebraic topology. Having learnt these basic concepts and their powerful techniques, one can then go on in any direction of the subject at an advanced level depending on one's interest and requirement.

Indeed, the main objective of this book is to introduce a student to some of the fundamental and important ideas of algebraic topology by emphasizing the relations of these ideas with other areas of mathematics. Rather than choosing the deep rooted ideas of modern topology, we concentrate our attention on introducing only as much algebraic machinery as necessary for problems we meet while dealing with homotopy theory, simplicial homology theory, singular homology theory, axiometic homology, differential topology , etc. in low dimensions. As a first course, we can and do hope that this approach is in better harmony with the historical development of the subject.

This book entitled with "Algebraic Topology and Applications" is designed as a text book for two semesters course for upper undergraduate and beginning graduate levels. The first one-semester course of this text is designed and is usually entitled 'Algebraic Topology' offered at the advanced undergraduate level and the second one-semester course that is usually entitled 'Advances in Algebraic Topology' offered at the beginning graduate level. It is accessible to junior mathematics majors who have studied general topology or point set topology. The second course is offered at the beginning of graduate course.

Pre-requisites

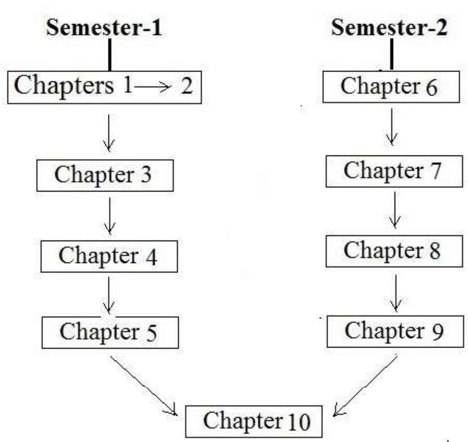
The essential prerequisites for reading this book are quite minimal: a basic understanding of fundamental notions of other topology, known as general topology or point set topology. This means that the reader should know what is meant by words like open, closed, compact, connected, and continuous and some of the basic facts about them. Because of our approach via analysis, the reader should know some basic facts about calculus, mainly for functions of one, two and multi-variables. In algebra, some basic notions of group, especially abelian group, and some basic linear algebra is required. Moreover, we have given complete definitions well explained by suitable examples, explanations, and proofs throughout the text, so that the book should be usable for indivisible study as well as for a course work.

Perhaps, in the later stage of study, we require the knowledge of advanced algebra, manifolds, complexes, combinatorics and differential geometry. It is very important because it opens the path to see the connection between different branches of mathematics. In fact, the development of topology over the last five decades has reached a high level in many directions. This process of development is still not finished, although a number of important problems that has faced by topologists were solved. The important feature of topology is that the very wide use of its methods in many of the other branches of modern mathematics such as those dealing with functional analysis, partial differential equations, classical mechanics, theoretical physics, general theory of relativity, mathematical economics, biology, etc. Topology is, indeed, an abstract mathematical theory that connects different branches, carries the potential of new development of mathematics. For example, Cauchy Integral Formula in complex analysis can be viewed differently in different branches of mathematics. From the viewpoint of partial differential equation, it is the solution of Dirichlet problem of Laplace equations, and from the viewpoint of harmonic analysis, it is the Abel sum of the Fourier series of a function. Nowadays algebraic topology has become a powerful instrument of mathematical research, and its language acquired importance.

The contents of the book has been divided into ten chapters, is devoted to the subject commonly called *algebraic topology* The first chapter is strictly preparatory and may be assigned as a self-study. It deals with basics of abstract algebra and point set topology reviewing group, group homomorphism, structure theorem, direct sum of abelian groups, free product of groups, free groups, modules, tensor product of modules and homomorphism, Euclidean spaces, continuous maps and product spaces, product topology, categories and functors. Chapter 2 devoted to pretty basic information on topological groups and transformation groups. This chapter is also concerned with the action of topological groups on topological spaces and associated orbit spaces. Chapter 3 deals with the fundamental group and covering spaces. In this chapter, the notion of fundamental group is introduced and its properties are studied. Some applications to algebra, vector field and index theory and mathematical economics are given. Chapter 4 studies on separation theorems in the plane which includes the Jordan separation theorem, invariance of domain, the Jordan curve theorem etc. and some application to complex analysis. Chapter 5 deals with the Seifert-van Kampen theorem which is a powerful theorem for computing the fundamental group of a path-connected space that can be decomposed as the union of two path-connected, open subsets whose intersection is also path-connected. The fundamental group of a wedge of circles, adjoining a two cells and the fundamental groups of the torus and the dunce cap are also given. Chapter 6 deals with simplicial complexes, simplexes, Triangulations and simplicial homology. In this chapter we also discuss oof simplicial complexes. Chapter 7 studies on homology, in particular, singular homotopy, topology invariance, Betti numbers and the Euler characteristic and computational homology. Chapter 8 deals with homology and cohomology theories. In this chapter, we discuss axiom for simplicial homology, relative homology, singular homology with coefficients, Mayer-Vietoris sequence for singular homology, singular cohomology and cohomology algebra, Jordan-Brouwer separation theorem, singular cohomology and cohomology algebra, and universal coefficient theorem for cohomology. Chapter 9 discusses classification of surfaces. Our study is focused on fundamental groups of surfaces and homology groups. Chapter 10 deals with some applications to calculus. In this chapter we discuss vector fields on surfaces, index theory for n-symmetric fields and some applications in computer graphics.

Suggested Course Outlines

As the book has been arranged according to the author's liking, one can observe that several topics are independent of one another, so it is profitable to advise the reader what should be read before a particular chapter in two semesters course. The dependencies of chapters are are marked by arrows as follows:



It may be observed that the book contains too much material that can be covered in a one-year course, but there is considerable flexibility for individual course design. Chapters $2-10$ are suitable for a full-year course in algebraic topology at the advanced undergraduate level/first year post graduate course as major or elective paper. There may be two options for two semesters course-Chapters $2-5$ for the first semester course and Chapters $6-10$ for the second semester course, or else Chapters $2-5$ followed by Chapter 10 are suitable for the first semester course and Chapters $6-9$ for the second semester course.

This book evolved from classes taught by the author in 5 Year Integrated M.Sc. at Center for Basic Sciences and Ph.D. course work at School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, India. Furthermore, the book is self-contained and the presentation is detailed, to avoid irritating readers by frequent references to details in other books. The examples are simple, to make the book teachable. We hope that this book will be extremely useful to students having background of 'abstract algebra' and 'point set topology'.

There are many people who deserve my gratitude in connection with the writing of this book. The author is highly indebted to many of his former teachers, colleagues, and students who directly or indirectly helped him in preparing this book. In particular, we extend very special cordial thanks to our colleagues and research collaborators Professors Satya Deo, V. Kannan, B. K. Sharma, M. Imdad, Q. H. Ansari, P. Veeramani, B. S. Thakur, D. R. Sahu, Hemant K. Nashine, P. P. Murthy all from India, Professors N. Shahzad and N. Hussain from Saudi Arabia, Professor S.S.Chang from China, Professors Shin Min Kang, J. S. Ume, B.S. Lee from S. Korea, Professor S. N. Mishra from S. Africa, Professor Brian Fisher from England, Professors Billy E. Rhoades, Ravi P. Agarwal, and Gerald Jungck from the USA, Professors Yeol Je Cho, Shin Min Kang from S. Korea who gave much to the beauty and power of mathematics. It is a pleasure to acknowledge the great help and technical support given us by the publisher, in their rapid and meticulous publication of the work.

April 22, 2022 H. K. Pathak
(On the occasion of World Earth Day)

Contents

About the Author

Hemant Kumar Pathak has been professor and Head of the School of Studies in Mathematics at the Pt. Ravishankar Shukla University, Raipur, India. He has been the dean of science, member of standing committee, chairman, UFM Committee, director of the Center for Basic Sciences, and director of the Human Resource Development Centre at the same university. He has previously worked at Kalyan P.G. College, Bhilai Nagar, and the Government Postgraduate College, Dhamtari, India. He has also been Vice Chancellor of Bharti Vishwavidyalaya, Durg. He earned his Ph.D. from Pt. Ravishankar Shukla University, Raipur in 1988.

Professor Pathak was awarded the "Distinguished Service Award 2011" by the Vijnana Parishad of India. With over 41 years of teaching and research experience, he has published three books, *Theory of Functions of Complex Variable* (Taylor & Fransis Group), *An Introduction to Nonlinear Analysis and Fixed Point Theory* (Springer Nature), *Complex Analysis and Applications* (Springer Nature), and more than 250 research papers in leading international journals of repute on approximation theory, operator theory, integration theory, fixed point theory, number theory, cryptography, summability theory, fuzzy set theory and Banach frames.

Professor Pathak currently serve on the editorial boards of American Journal of Computational and Applied Mathematics, Fixed Point Theory and Algorithms for Sciences and Engineering (Springer Nature), and the Journal of Modern Methods in Numerical Mathematics, and as a reviewer for the Mathematical Review of the American Mathematical Society. In addition, he is a life member of the Allahabad Mathematical Society, Bharata Ganita parishad, the Vijana Parishad of India, Calcutta Mathematical Society, and National Academy of Mathematics.

Glossary of Symbols

$\mathbb{N}$ or $\mathbb{Z}_+$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{C}_\infty$	extended complex plane $\mathbb{C} \cup \{\infty\}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}^n$	n-dimensional unitary space
$\mathbb{H}^n$	The set of all n-tuples $(q_1, q_2, \cdots, q_n)$ of quaternions
$\in$	belongs to
$\notin$	does not belong to
$\vert, :$	such that
$\subset$	subset
$\cup$	union
$\cap$	intersection
$-, \setminus$	difference
I, Λ	index set
$\emptyset$	empty set
$\prod_{i=1}^{n}$	product, i ranging from 1 to n
(a,b)	ordered pair
int A, A°	interior of A
∂A, A^b	boundary of A
$A \times B$	cartesian product of sets A and B
$\mathcal{D}(f)$	domain of f
$\mathcal{R}(f)$	range of f
aRb	a is R-related to b
$\forall$	for all
$\Rightarrow$	implies
$f: X \to Y$	f is a function from X to Y
$f(A)$	image of A under f
$f^{-1}(B)$	inverse image of B under f
i_X	identity map on X
$i: B \to A$	inclusion function
$f\vert_A$	restriction f to A
$(x_n), \{x_n\}$	sequence whose n^{th} term is x_n

$\mathscr{P}(X)$	collection of all subsets of X
(a,b)	open interval $\{x : a < x < b\}$
$[a,b]$	closed interval $\{x : a \leq x \leq b\}$
f^{-1}	inverse mapping
$f \circ g$	composition of mappings f and g
d, ρ	metrics
$\bar{\rho}$	uniform metric
$B(x,r)$	open ball centered at x and of radius r
$\overline{B}(x,r)$	closed ball centered at x and of radius r
$\mathcal{T}$	topology
$\mathcal{T}_d$	topology induced by metric d
$\mathcal{T}_p$	topology of pointwise convergence
$\mathcal{T}_c$	topology of compact convergence (or compact-open topology)
$\mathcal{B}$	base for a topology
$\pi_n(X, x_0)$	n-dimensional homotopy group
$\mathbb{R}P^n$	real n-dimensional projective space
$\sup S$	supremum of S
$\inf S$	infimum of S
$\lvert z \rvert$	modulus of z
$\bar{z}$	conjugate of z
$P_n(z)$	polynomial of degree n
$\exp(z)$	exponential function of z
$(x : P(x))$	set of all x such that $P(x)$
$\bigoplus_{\alpha \in J}$	direct sum of groups $G_\alpha,\ \alpha \in J$
$\prod_{\alpha \in J} X_\alpha$	direct product of spaces X_α
$\prod^{*}_{\alpha \in J} G_\alpha$	direct product of free groups G_α